New classes of modified teleparallel gravity models

Sebastián Bahamonde

sebastian.beltran.14@ucl.ac.uk
Department of Mathematics, University College London.

Geometric Foundations of Gravity in Tartu.

Work in collaboration with Christian Böhmer and Martin Krššák. Based on arXiv:1706.04920 [gr-qc]

31st August 2017

Outline

Outline

Teleparallel gravity models

- In teleparallel gravity, the connection is assumed to satisfy the condition of zero curvature

$$R^{a}{}_{b\mu\nu}(\omega^{a}{}_{b\mu}) = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu} \equiv 0\,, \tag{1}$$

which is solved by the pure gauge-like connection given by

$$\omega^{a}{}_{b\mu} = \Lambda^{a}{}_{c}\partial_{\mu}\Lambda_{b}{}^{c}\,, \tag{2}$$

where $\Lambda_{b}{}^{c} = (\Lambda^{-1})^{c}{}_{b}$

Teleparallel gravity models

- The torsion tensor of this connection

$$T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu}) = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu \,, \quad (3)$$

is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

- It can be decomposed in three irreducible parts:

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu) + \epsilon_{\lambda\mu\nu\rho} a^\rho \,, \quad (4)$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu} \,, \quad a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho} T^{\nu\sigma\rho} \,, \quad (5)$$

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda} v_\mu + g_{\nu\mu} v_\lambda) - \frac{1}{3} g_{\lambda\mu} v_\nu \,. \quad (6)$$

Teleparallel gravity models

- The torsion tensor of this connection

$$T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu}) = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu \,, \quad (3)$$

is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

- It can be decomposed in three irreducible parts:

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu) + \epsilon_{\lambda\mu\nu\rho} a^\rho \,, \quad (4)$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu} \,, \qquad a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho} T^{\nu\sigma\rho} \,, \quad (5)$$

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda} v_\mu + g_{\nu\mu} v_\lambda) - \frac{1}{3} g_{\lambda\mu} v_\nu \,. \quad (6)$$

Teleparallel models - TEGR

- The teleparallel equivalent of general relativity (TEGR) is constructed with

Standard Teleparallel gravity Lagrangian

$$\mathcal{L}_{\mathrm{TEGR}} = \frac{e}{2\kappa} T \,, \tag{7}$$

where $e = \det(e_a{}^\mu)$ and T is the scalar torsion defined as

$$T = \frac{3}{2} T_{\mathrm{ax}} + \frac{2}{3} T_{\mathrm{ten}} - \frac{2}{3} T_{\mathrm{vec}} \,, \tag{8}$$

$$T_{\mathrm{ten}} = t_{ijk} t^{ijk} = \frac{1}{2}\left(T_{ijk}T^{ijk} + T_{ijk}T^{jik}\right) - \frac{1}{2} T_i T^i \,, \tag{9}$$

$$T_{\mathrm{ax}} = a_i a^i = \frac{1}{18}\left(T_{ijk}T^{ijk} - 2T_{ijk}T^{jik}\right) \,, \tag{10}$$

$$T_{\mathrm{vec}} = v_i v^i = T_i T^i \,. \tag{11}$$

Teleparallel models - TEGR

- The teleparallel equivalent of general relativity (TEGR) is constructed with

Standard Teleparallel gravity Lagrangian

$$\mathcal{L}_{\rm TEGR} = \frac{e}{2\kappa} T \,, \tag{7}$$

where $e = \det(e_a{}^\mu)$ and T is the scalar torsion defined as

$$T = \frac{3}{2} T_{\rm ax} + \frac{2}{3} T_{\rm ten} - \frac{2}{3} T_{\rm vec} \,, \tag{8}$$

$$T_{\rm ten} = t_{ijk} t^{ijk} = \frac{1}{2}\left(T_{ijk}T^{ijk} + T_{ijk}T^{jik}\right) - \frac{1}{2} T_i T^i \,, \tag{9}$$

$$T_{\rm ax} = a_i a^i = \frac{1}{18}\left(T_{ijk}T^{ijk} - 2T_{ijk}T^{jik}\right) \,, \tag{10}$$

$$T_{\rm vec} = v_i v^i = T_i T^i \,. \tag{11}$$

Teleparallel models - TEGR

- The teleparallel equivalent of general relativity (TEGR) is constructed with

Standard Teleparallel gravity Lagrangian

$$\mathcal{L}_{\rm TEGR} = \frac{e}{2\kappa} T\,, \tag{7}$$

where $e = \det(e_a{}^{\mu})$ and T is the scalar torsion defined as

$$T = \frac{3}{2} T_{\rm ax} + \frac{2}{3} T_{\rm ten} - \frac{2}{3} T_{\rm vec}\,, \tag{8}$$

$$T_{\rm ten} = t_{ijk} t^{ijk} = \frac{1}{2}\left(T_{ijk} T^{ijk} + T_{ijk} T^{jik}\right) - \frac{1}{2} T_i T^i\,, \tag{9}$$

$$T_{\rm ax} = a_i a^i = \frac{1}{18}\left(T_{ijk} T^{ijk} - 2 T_{ijk} T^{jik}\right)\,, \tag{10}$$

$$T_{\rm vec} = v_i v^i = T_i T^i\,. \tag{11}$$

Teleparallel models - TEGR

- The teleparallel equivalent of general relativity (TEGR) is constructed with

Standard Teleparallel gravity Lagrangian

$$\mathcal{L}_{\mathrm{TEGR}} = \frac{e}{2\kappa} T \,, \tag{7}$$

where $e = \det(e_a{}^\mu)$ and T is the scalar torsion defined as

$$T = \frac{3}{2} T_{\mathrm{ax}} + \frac{2}{3} T_{\mathrm{ten}} - \frac{2}{3} T_{\mathrm{vec}} \,, \tag{8}$$

$$T_{\mathrm{ten}} = t_{ijk} t^{ijk} = \frac{1}{2} \left(T_{ijk} T^{ijk} + T_{ijk} T^{jik} \right) - \frac{1}{2} T_i T^i \,, \tag{9}$$

$$T_{\mathrm{ax}} = a_i a^i = \frac{1}{18} \left(T_{ijk} T^{ijk} - 2 T_{ijk} T^{jik} \right) \,, \tag{10}$$

$$T_{\mathrm{vec}} = v_i v^i = T_i T^i \,. \tag{11}$$

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\mathrm{NGR}} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\mathrm{ax}} + a_2 T_{\mathrm{ten}} + a_3 T_{\mathrm{vec}}\Big)\,. \qquad (12)$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\mathrm{NGR}} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\mathrm{ax}} + a_2 T_{\mathrm{ten}} + a_3 T_{\mathrm{vec}}\Big)\,. \tag{12}$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\rm NGR} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}\Big)\,. \tag{12}$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\rm NGR} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}\Big)\,. \qquad (12)$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\mathrm{NGR}} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\mathrm{ax}} + a_2 T_{\mathrm{ten}} + a_3 T_{\mathrm{vec}}\Big)\,. \qquad (12)$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

$f(T)$ gravity Lagrangian (Bengochea and Ferraro, 2008)

$$\mathcal{L}_{f(T)} = \frac{e}{2\kappa} f(T)\,. \qquad (13)$$

Teleparallel models - some famous modifications

- The first generalisation of this theory was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\rm NGR} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}\Big)\,. \tag{12}$$

where a_i are constants. Clearly if $a_1 = 3/2, a_2 = 2/3$ and $a_3 = -2/3$ we recover TEGR.

- In the recent decade, another straightforward generalization of the TEGR Lagrangian is

$f(T)$ gravity Lagrangian (Bengochea and Ferraro, 2008)

$$\mathcal{L}_{f(T)} = \frac{e}{2\kappa} f(T)\,. \tag{13}$$

Teleparallel models - some famous modifications

- Another interesting model based on conformal gravity models is the following

Conformal teleparallel gravity (Maluf and Faria, 2011)

$$\mathcal{L}_{\rm CTG} = \frac{e}{2\kappa}\tilde{T}^2\,, \tag{14}$$

where $\tilde{T}$ is the torsion scalar taken from the new general relativity Lagrangian with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. This theory is invariant under conformal transformations.

- Our aim is to construct new classes of teleparallel theories of gravity which involves all of those models and study its properties.

Teleparallel models - some famous modifications

- Another interesting model based on conformal gravity models is the following

Conformal teleparallel gravity (Maluf and Faria, 2011)

$$\mathcal{L}_{\mathrm{CTG}} = \frac{e}{2\kappa}\tilde{T}^2 \,, \tag{14}$$

where $\tilde{T}$ is the torsion scalar taken from the new general relativity Lagrangian with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. This theory is invariant under conformal transformations.

- Our aim is to construct new classes of teleparallel theories of gravity which involves all of those models and study its properties.

Teleparallel models - some famous modifications

- Another interesting model based on conformal gravity models is the following

Conformal teleparallel gravity (Maluf and Faria, 2011)

$$\mathcal{L}_{\rm CTG} = \frac{e}{2\kappa}\tilde{T}^2\,, \tag{14}$$

where $\tilde{T}$ is the torsion scalar taken from the new general relativity Lagrangian with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. This theory is invariant under conformal transformations.

- Our aim is to construct new classes of teleparallel theories of gravity which involves all of those models and study its properties.

Teleparallel models - some famous modifications

- Another interesting model based on conformal gravity models is the following

Conformal teleparallel gravity (Maluf and Faria, 2011)

$$\mathcal{L}_{\mathrm{CTG}} = \frac{e}{2\kappa}\tilde{T}^2\,, \tag{14}$$

where $\tilde{T}$ is the torsion scalar taken from the new general relativity Lagrangian with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. This theory is invariant under conformal transformations.

- Our aim is to construct new classes of teleparallel theories of gravity which involves all of those models and study its properties.

Teleparallel models - some famous modifications

- Another interesting model based on conformal gravity models is the following

Conformal teleparallel gravity (Maluf and Faria, 2011)

$$\mathcal{L}_{\rm CTG} = \frac{e}{2\kappa}\tilde{T}^2\,, \tag{14}$$

where $\tilde{T}$ is the torsion scalar taken from the new general relativity Lagrangian with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. This theory is invariant under conformal transformations.

- Our aim is to construct new classes of teleparallel theories of gravity which involves all of those models and study its properties.

Outline

New class of modified Teleparallel gravity

- Motivated by all the models mentioned before, let us consider the following model,

Lagrangian of the model

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}) + \mathcal{L}_M \,, \quad (15)$$

- This model includes all previous models.
- Since the torsion pieces only contain first partial derivatives of the tetrad, the resulting field equations will be of second order.

New class of modified Teleparallel gravity

- Motivated by all the models mentioned before, let us consider the following model,

Lagrangian of the model

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\text{ax}}, T_{\text{ten}}, T_{\text{vec}}) + \mathcal{L}_M \,, \tag{15}$$

- This model includes all previous models.
- Since the torsion pieces only contain first partial derivatives of the tetrad, the resulting field equations will be of second order.

New class of modified Teleparallel gravity

- Motivated by all the models mentioned before, let us consider the following model,

Lagrangian of the model

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}) + \mathcal{L}_M \,, \tag{15}$$

- This model includes all previous models.
- Since the torsion pieces only contain first partial derivatives of the tetrad, the resulting field equations will be of second order.

New class of modified Teleparallel gravity

- Motivated by all the models mentioned before, let us consider the following model,

Lagrangian of the model

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}) + \mathcal{L}_M \,, \tag{15}$$

- This model includes all previous models.
- Since the torsion pieces only contain first partial derivatives of the tetrad, the resulting field equations will be of second order.

Inclusion of parity violating terms and higher-order invariants

- The three invariants considered are the most general, quadratic, parity preserving, irreducible torsion invariants.
- If we relax the requirement of parity preservation, we have two new quadratic parity violating invariants which are

$$P_1 = v^{\mu} a_{\mu} \,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_{\lambda}{}^{\rho\sigma} \,. \tag{16}$$

- We can then naturally consider a straightforward generalization:

New class with parity violating terms

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, P_1, P_2) \,. \tag{17}$$

- The Lagrangian (17) is the most general Lagrangian taken as a function of all invariants quadratic in torsion.

Inclusion of parity violating terms and higher-order invariants

- The three invariants considered are the most general, quadratic, parity preserving, irreducible torsion invariants.
- If we relax the requirement of parity preservation, we have two new quadratic parity violating invariants which are

$$P_1 = v^{\mu} a_{\mu}\,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_{\lambda}{}^{\rho\sigma}\,. \tag{16}$$

- We can then naturally consider a straightforward generalization:

New class with parity violating terms

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, P_1, P_2)\,. \tag{17}$$

- The Lagrangian (17) is the most general Lagrangian taken as a function of all invariants quadratic in torsion.

Inclusion of parity violating terms and higher-order invariants

- The three invariants considered are the most general, quadratic, parity preserving, irreducible torsion invariants.
- If we relax the requirement of parity preservation, we have two new quadratic parity violating invariants which are

$$P_1 = v^\mu a_\mu \,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\rho\sigma} \,. \tag{16}$$

- We can then naturally consider a straightforward generalization:

New class with parity violating terms

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, P_1, P_2) \,. \tag{17}$$

- The Lagrangian (17) is the most general Lagrangian taken as a function of all invariants quadratic in torsion.

Inclusion of parity violating terms and higher-order invariants

- The three invariants considered are the most general, quadratic, parity preserving, irreducible torsion invariants.
- If we relax the requirement of parity preservation, we have two new quadratic parity violating invariants which are

$$P_1 = v^{\mu} a_{\mu}\,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_{\lambda}{}^{\rho\sigma}\,. \tag{16}$$

- We can then naturally consider a straightforward generalization:

New class with parity violating terms

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, P_1, P_2)\,. \tag{17}$$

- The Lagrangian (17) is the most general Lagrangian taken as a function of all invariants quadratic in torsion.

Inclusion of parity violating terms and higher-order invariants

- The three invariants considered are the most general, quadratic, parity preserving, irreducible torsion invariants.
- If we relax the requirement of parity preservation, we have two new quadratic parity violating invariants which are

$$P_1 = v^\mu a_\mu \,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma} t^{\lambda\mu\nu} t_\lambda{}^{\rho\sigma} \,. \tag{16}$$

- We can then naturally consider a straightforward generalization:

New class with parity violating terms

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, P_1, P_2) \,. \tag{17}$$

- The Lagrangian (17) is the most general Lagrangian taken as a function of all invariants quadratic in torsion.

Inclusion of the boundary term

- One can also consider derivatives of torsion which creates higher-order field equations.
- One well-motivated term is the so-called boundary term

$$B = \frac{2}{e}\partial_\mu(ev^\mu)\,, \tag{18}$$

which relates the torsion scalar and the Ricci scalar via

Relationship between Ricci scalar and Torsion scalar

$$R = -T + \frac{2}{e}\partial_\mu(ev^\mu) = -T + B\,. \tag{19}$$

Inclusion of the boundary term

- One can also consider derivatives of torsion which creates higher-order field equations.
- One well-motivated term is the so-called boundary term

$$B = \frac{2}{e}\partial_\mu(ev^\mu)\,, \tag{18}$$

which relates the torsion scalar and the Ricci scalar via

Relationship between Ricci scalar and Torsion scalar

$$R = -T + \frac{2}{e}\partial_\mu(ev^\mu) = -T + B\,. \tag{19}$$

Inclusion of the boundary term

- One can also consider derivatives of torsion which creates higher-order field equations.
- One well-motivated term is the so-called boundary term

$$B = \frac{2}{e}\partial_\mu(ev^\mu)\,, \tag{18}$$

which relates the torsion scalar and the Ricci scalar via

Relationship between Ricci scalar and Torsion scalar

$$R = -T + \frac{2}{e}\partial_\mu(ev^\mu) = -T + B\,. \tag{19}$$

Inclusion of the boundary term

- We can then include the boundary term and consider the Lagrangian

New class of modified TG with a boundary term

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B)\,. \tag{20}$$

- This theory is a 4th order theory and one can recover $f(R)$ gravity directly by choosing $f = f(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B) = f(R)$.

Inclusion of the boundary term

- We can then include the boundary term and consider the Lagrangian

New class of modified TG with a boundary term

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B)\,. \tag{20}$$

- This theory is a 4th order theory and one can recover $f(R)$ gravity directly by choosing $f = f(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B) = f(R)$.

Inclusion of the boundary term

- We can then include the boundary term and consider the Lagrangian

New class of modified TG with a boundary term

$$\mathcal{L} = \frac{e}{2\kappa} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B)\,. \tag{20}$$

- This theory is a 4th order theory and one can recover $f(R)$ gravity directly by choosing $f = f(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B) = f(R)$.

Conformal transformations

- Let us rewrite the action by introducing two sets of four auxiliary fields $\phi_A = \{T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B\}$ and χ_A:

$$S = \frac{1}{2\kappa}\int \Big[f(\phi_A) + \chi_1(T_{\rm ax} - \phi_1) + \chi_2(T_{\rm ten} - \phi_2) + \chi_3(T_{\rm vec} - \phi_3) + \chi_4(B - \phi_4)\Big]\, e\, d^4x\,.$$

- Hence, this action can be rewritten as

$$S = \frac{1}{2\kappa}\int \Big[\sum_{B=1}^{4} F_B(\phi_A)\phi_B - V(\phi_A)\Big]\, e\, d^4x\,,$$

where we have defined the energy potential as

$$\chi_A = \frac{\partial f(\phi_B)}{\partial \phi_A} := F_A\,, \quad V(\phi_A) = \sum_{B=1}^{4}\phi_B F_B - f(\phi_A)\,.$$

Conformal transformations

- Let us rewrite the action by introducing two sets of four auxiliary fields $\phi_A = \{T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B\}$ and χ_A:

$$S = \frac{1}{2\kappa}\int \Big[f(\phi_A) + \chi_1(T_{\rm ax} - \phi_1) + \chi_2(T_{\rm ten} - \phi_2) + \chi_3(T_{\rm vec} - \phi_3) + \chi_4(B - \phi_4)\Big]\, e\, d^4x \,.$$

- Hence, this action can be rewritten as

$$S = \frac{1}{2\kappa}\int \Big[\sum_{B=1}^{4} F_B(\phi_A)\phi_B - V(\phi_A)\Big]\, e\, d^4x \,,$$

where we have defined the energy potential as

$$\chi_A = \frac{\partial f(\phi_B)}{\partial \phi_A} := F_A \,, \quad V(\phi_A) = \sum_{B=1}^{4} \phi_B F_B - f(\phi_A) \,.$$

Conformal transformations

- Next, let us apply a conformal transformation to the metric

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}\,. \tag{21}$$

- The important quantities changes as follows

$$T_{\rm ax} = \Omega^2 \hat{T}_{\rm ax}\,, \tag{22}$$

$$T_{\rm ten} = \Omega^2 \hat{T}_{\rm ten}\,, \tag{23}$$

$$T_{\rm vec} = \Omega^2 \hat{T}_{\rm vec} + 6\Omega \hat{T}^{\mu}\hat{\partial}_{\mu}\Omega + 9\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega)\,, \tag{24}$$

$$B = \Omega^2 \hat{B} - 4\Omega \hat{T}^{\mu}\hat{\partial}_{\mu}\Omega - 18\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}\Omega + \frac{6}{\hat{e}}\Omega\hat{\partial}_{\mu}(\hat{e}\hat{g}^{\mu\nu}\hat{\partial}_{\nu}\Omega)\,. \tag{25}$$

- By transforming all the important quantities, we arrive at

$$S = \frac{1}{2\kappa}\int \Big[F_1(\phi_A)\Omega^{-2}\hat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\hat{T}_{\rm ten} + F_3(\phi_A)\Omega^{-2}\hat{T}_{\rm vec}$$
$$+ 2\Omega^{-2}\hat{T}^{\mu}\Big(3F_3(\phi_A)\Omega^{-1}\hat{\partial}_{\mu}\Omega - \partial_{\mu}F_4(\phi_A)\Big) + 9F_3(\phi_A)\Omega^{-4}\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega)$$
$$+ -6\Omega^{-3}(\partial^{\mu}\Omega)\partial_{\mu}F_4(\phi_A) - \Omega^{-4}V(\phi_A)\Big]\,\hat{e}\,d^4x\,. \tag{26}$$

Conformal transformations

- Next, let us apply a conformal transformation to the metric

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}\,. \tag{21}$$

- The important quantities changes as follows

$$T_{\rm ax} = \Omega^2 \hat{T}_{\rm ax}\,, \tag{22}$$

$$T_{\rm ten} = \Omega^2 \hat{T}_{\rm ten}\,, \tag{23}$$

$$T_{\rm vec} = \Omega^2 \hat{T}_{\rm vec} + 6\Omega \hat{T}^\mu \hat{\partial}_\mu \Omega + 9\hat{g}^{\mu\nu}(\hat{\partial}_\mu \Omega)(\hat{\partial}_\nu \Omega)\,, \tag{24}$$

$$B = \Omega^2 \hat{B} - 4\Omega \hat{T}^\mu \hat{\partial}_\mu \Omega - 18\hat{\partial}^\mu \Omega \hat{\partial}_\mu \Omega + \frac{6}{\hat{e}}\Omega \hat{\partial}_\mu (\hat{e}\hat{g}^{\mu\nu}\hat{\partial}_\nu \Omega)\,. \tag{25}$$

- By transforming all the important quantities, we arrive at

$$S = \frac{1}{2\kappa}\int \Big[F_1(\phi_A)\Omega^{-2}\hat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\hat{T}_{\rm ten} + F_3(\phi_A)\Omega^{-2}\hat{T}_{\rm vec}$$
$$+ 2\Omega^{-2}\hat{T}^\mu\Big(3F_3(\phi_A)\Omega^{-1}\hat{\partial}_\mu\Omega - \partial_\mu F_4(\phi_A)\Big) + 9F_3(\phi_A)\Omega^{-4}\hat{g}^{\mu\nu}(\hat{\partial}_\mu\Omega)(\hat{\partial}_\nu\Omega)$$
$$+ -6\Omega^{-3}(\partial^\mu\Omega)\partial_\mu F_4(\phi_A) - \Omega^{-4}V(\phi_A)\Big]\,\hat{e}\,d^4x\,. \tag{26}$$

Conformal transformations

- Next, let us apply a conformal transformation to the metric

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}\,. \tag{21}$$

- The important quantities changes as follows

$$T_{\rm ax} = \Omega^2 \hat{T}_{\rm ax}\,, \tag{22}$$

$$T_{\rm ten} = \Omega^2 \hat{T}_{\rm ten}\,, \tag{23}$$

$$T_{\rm vec} = \Omega^2 \hat{T}_{\rm vec} + 6\Omega \hat{T}^{\mu}\hat{\partial}_{\mu}\Omega + 9\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega)\,, \tag{24}$$

$$B = \Omega^2 \hat{B} - 4\Omega \hat{T}^{\mu}\hat{\partial}_{\mu}\Omega - 18\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}\Omega + \frac{6}{\hat{e}}\Omega\hat{\partial}_{\mu}(\hat{e}\hat{g}^{\mu\nu}\hat{\partial}_{\nu}\Omega)\,. \tag{25}$$

- By transforming all the important quantities, we arrive at

$$S = \frac{1}{2\kappa}\int \Big[F_1(\phi_A)\Omega^{-2}\hat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\hat{T}_{\rm ten} + F_3(\phi_A)\Omega^{-2}\hat{T}_{\rm vec} + 2\Omega^{-2}\hat{T}^{\mu}\Big(3F_3(\phi_A)\Omega^{-1}\hat{\partial}_{\mu}\Omega - \partial_{\mu}F_4(\phi_A)\Big) + 9F_3(\phi_A)\Omega^{-4}\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega) + -6\Omega^{-3}(\partial^{\mu}\Omega)\partial_{\mu}F_4(\phi_A) - \Omega^{-4}V(\phi_A)\Big]\,\hat{e}\,d^4x\,. \tag{26}$$

Conformal transformations

- Next, let us apply a conformal transformation to the metric

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}\,. \tag{21}$$

- The important quantities changes as follows

$$T_{\rm ax} = \Omega^2 \hat{T}_{\rm ax}\,, \tag{22}$$

$$T_{\rm ten} = \Omega^2 \hat{T}_{\rm ten}\,, \tag{23}$$

$$T_{\rm vec} = \Omega^2 \hat{T}_{\rm vec} + 6\Omega \hat{T}^{\mu} \hat{\partial}_{\mu}\Omega + 9\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega)\,, \tag{24}$$

$$B = \Omega^2 \hat{B} - 4\Omega \hat{T}^{\mu}\hat{\partial}_{\mu}\Omega - 18\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}\Omega + \frac{6}{\hat{e}}\Omega\hat{\partial}_{\mu}(\hat{e}\hat{g}^{\mu\nu}\hat{\partial}_{\nu}\Omega)\,. \tag{25}$$

- By transforming all the important quantities, we arrive at

$$S = \frac{1}{2\kappa}\int \Big[F_1(\phi_A)\Omega^{-2}\hat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\hat{T}_{\rm ten} + F_3(\phi_A)\Omega^{-2}\hat{T}_{\rm vec}$$
$$+ 2\Omega^{-2}\hat{T}^{\mu}\Big(3F_3(\phi_A)\Omega^{-1}\hat{\partial}_{\mu}\Omega - \partial_{\mu}F_4(\phi_A)\Big) + 9F_3(\phi_A)\Omega^{-4}\hat{g}^{\mu\nu}(\hat{\partial}_{\mu}\Omega)(\hat{\partial}_{\nu}\Omega)$$
$$+ -6\Omega^{-3}(\partial^{\mu}\Omega)\partial_{\mu}F_4(\phi_A) - \Omega^{-4}V(\phi_A)\Big]\,\hat{e}\,d^4x\,. \tag{26}$$

Minimal and non-minimal couplings

- To eliminate all the couplings between the scalar field and $\hat{T}^\mu$ (or equivalently B), the function must satisfy

$$f^{(0,0,1,1)}(\phi_A)^2 = f^{(0,0,0,2)}f^{(0,0,2,0)}(\phi_A)\,, \tag{27}$$

$$f^{(0,1,1,0)}(\phi_A)f^{(0,0,0,2)}(\phi_A) = f^{(0,0,1,1)}(\phi_A)f^{(0,1,0,1)}(\phi_A)\,, \tag{28}$$

$$f^{(1,0,1,0)}(\phi_A)f^{(0,1,0,1)}(\phi_A) = f^{(0,1,1,0)}(\phi_A)f^{(1,0,0,1)}(\phi_A)\,, \tag{29}$$

  which gives us a non-minimally coupled theory between T_i and a scalar field.

Minimal and non-minimal couplings

- To obtain a theory where the scalar field is minimally coupled with the torsion scalar (an Einstein frame) we must ALSO impose

$$\Omega^2 = -\frac{2}{3}F_1(\phi_A) = -\frac{3}{2}F_2(\phi_A) = \frac{3}{2}F_3(\phi_A)\,. \tag{30}$$

- The unique theory (besides TEGR) which satisfies this condition plus the conditions mentioned in the previous slide is

Unique theory with an Einstein frame

$$f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) = \tilde{f}\Big(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B\Big) = \tilde{f}(R)\,. \tag{31}$$

Minimal and non-minimal couplings

- To obtain a theory where the scalar field is minimally coupled with the torsion scalar (an Einstein frame) we must ALSO impose

$$\Omega^2 = -\frac{2}{3}F_1(\phi_A) = -\frac{3}{2}F_2(\phi_A) = \frac{3}{2}F_3(\phi_A)\,. \qquad (30)$$

- The unique theory (besides TEGR) which satisfies this condition plus the conditions mentioned in the previous slide is

Unique theory with an Einstein frame

$$f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) = \tilde{f}\Big(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B\Big) = \tilde{f}(R)\,. \qquad (31)$$

Minimal and non-minimal couplings

- To obtain a theory where the scalar field is minimally coupled with the torsion scalar (an Einstein frame) we must ALSO impose

$$\Omega^2 = -\frac{2}{3}F_1(\phi_A) = -\frac{3}{2}F_2(\phi_A) = \frac{3}{2}F_3(\phi_A)\,. \tag{30}$$

- The unique theory (besides TEGR) which satisfies this condition plus the conditions mentioned in the previous slide is

Unique theory with an Einstein frame

$$f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) = \tilde{f}\Big(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B\Big) = \tilde{f}(R)\,. \tag{31}$$

Local Lorentz transformations

- The violation of local Lorentz invariance is a consequence of choosing a particular frame in which the spin connection vanishes

$$\omega^{a}{}_{b\mu} = 0\,. \tag{32}$$

  The origin of this approach can be traced back to Einstein's teleparallel theory.
- The torsion tensor with zero connection is no longer a tensor under local Lorentz transformations.

Local Lorentz transformations

- The violation of local Lorentz invariance is a consequence of choosing a particular frame in which the spin connection vanishes

$$\omega^{a}{}_{b\mu} = 0\,. \tag{32}$$

  The origin of this approach can be traced back to Einstein's teleparallel theory.

- The torsion tensor with zero connection is no longer a tensor under local Lorentz transformations.

Local Lorentz transformations

- The fully covariant approach to teleparallel gravity is based on a generalization of the teleparallel connection satisfying the zero curvature connection $\rightarrow$ spin connection $\neq$ zero

Standard TG	Covariant TG
$w^a{}_{b\mu} = 0$	$w^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c$
Non invariant LT	Invariant LT
Alleviating: Good tetrads	Problem: field without dynamic

Local Lorentz transformations

- The fully covariant approach to teleparallel gravity is based on a generalization of the teleparallel connection satisfying the zero curvature connection $\rightarrow$ spin connection $\neq$ zero

Standard TG	Covariant TG
$w^a{}_{b\mu} = 0$	$w^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c$
Non invariant LT	Invariant LT
Alleviating: Good tetrads	Problem: field without dynamic

Local Lorentz transformations

- The fully covariant approach to teleparallel gravity is based on a generalization of the teleparallel connection satisfying the zero curvature connection $\rightarrow$ spin connection $\neq$ zero

Standard TG	Covariant TG
$w^{a}{}_{b\mu} = 0$	$w^{a}{}_{b\mu} = \Lambda^{a}{}_{c}\partial_{\mu}\Lambda_{b}{}^{c}$
Non invariant LT	Invariant LT
Alleviating: Good tetrads	Problem: field without dynamic

Outline

How are all those models connected?

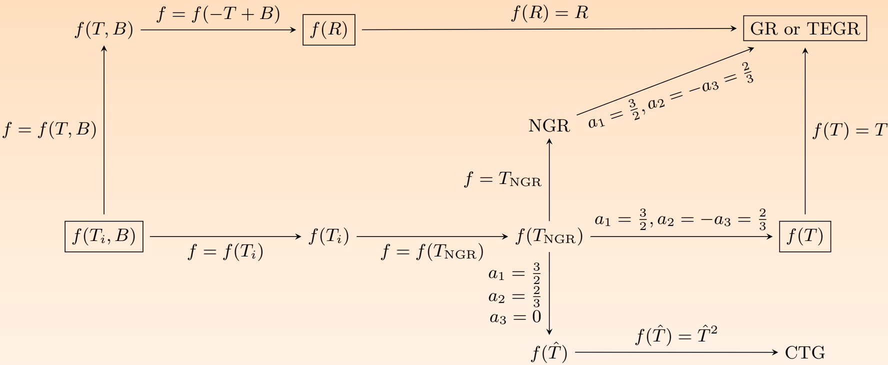

Figure: Relationship between different modified gravity models and General Relativity where $T_i=(T_{\rm ax},T_{\rm ten},T_{\rm vec})$, $T_{\rm NGR}=a_1T_{\rm ax}+a_2T_{\rm ten}+a_3T_{\rm vec}$ and $\tilde{T}=\frac{3}{2}T_{\rm ax}+\frac{2}{3}T_{\rm ten}$. The abbreviations NGR, CTG and TEGR mean new general relativity, teleparallel conformal gravity and teleparallel equivalent of general relativity respectively.

Conclusions

- We formulated the most general (well-motivated) 2nd order teleparallel theory of gravity, based on the squares of the irreducible pieces of the torsion tensor $T_{\rm ax}, T_{\rm ten}$ and $T_{\rm vec}$.
- All the most important teleparallel theories can be recovered from this approach.
- We demonstrated that the unique theory which have an Einstein frame is either $f(-T+B) = f(R)$ or standard TEGR.

Conclusions

- We formulated the most general (well-motivated) 2nd order teleparallel theory of gravity, based on the squares of the irreducible pieces of the torsion tensor $T_{\rm ax}, T_{\rm ten}$ and $T_{\rm vec}$.
- All the most important teleparallel theories can be recovered from this approach.
- We demonstrated that the unique theory which have an Einstein frame is either $f(-T+B)=f(R)$ or standard TEGR.

Conclusions

- We formulated the most general (well-motivated) 2nd order teleparallel theory of gravity, based on the squares of the irreducible pieces of the torsion tensor $T_{\rm ax}, T_{\rm ten}$ and $T_{\rm vec}$.
- All the most important teleparallel theories can be recovered from this approach.
- We demonstrated that the unique theory which have an Einstein frame is either $f(-T+B)=f(R)$ or standard TEGR.